

# Quantum Knizhnik-Zamolodchikov equations of level 0 and form factors in SOS model

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## Abstract

The cyclic SOS model is considered on the basis of Smirnov's form factor bootstrap approach. Integral solutions to the quantum Knizhnik-Zamolodchikov equations of level 0 are presented.

## 1 Intorduction

In a previous paper [1] we presented integral formulae for correlation functions of both cyclic SOS and antiferromagnetic XYZ models, by directly solving the bootstrap equations. The bootstrap equations, the master equations of correlation functions in integrable models, can be derived on the basis of the CTM bootstrap [2]. In the present paper, we are interested in form factors of the elliptic integrable model. Form factors are very important objects, because all physical quantities can be expressed in terms of form factors, in principle.

Through the study of form factors of the sine-Gordon model, Smirnov [3] found three axioms as sufficient conditions for the local commutativity of local fields in the model. These three axioms consist of (i)  $S$ -matrix symmetry, (ii) the cyclicity condition, and (iii) the annihilation pole condition. Furthermore, Smirnov observed in Ref. [4] that the first two axioms imply the quantum Knizhnik-Zamolodchikov equation [5] of level 0.

Form factors were originally defined as matrix elements of local operators. However, we refer to the objects that satisfy Smirnov's three axioms as 'form factors'. In this sense, we wish to construct integral formulae for cyclic SOS form factors in this paper. This is a preliminary study, in which we make preparations to construct those of the XYZ form factors.

There are several ways to address the problem of constructing form factors of integrable models. In the vertex operator approach, [6, 7] form factors can be constructed as traces of products of type II vertex operators and some local operators. Lashkevich [8] used the vertex-face correspondence [9] to construct a free field representation of the type II vertex operators of the XYZ model in terms of

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those of the eight-vertex SOS model [10, 11], using an insertion of a non-local tail operator. The type I vertex operators of the XYZ model were constructed in Ref. [12]. These operators are relevant for the expression of the local operators. Shiraishi [13] directly constructed the type II and type I vertex operators of the eight-vertex model as intertwiners of the  $q$ -deformed Virasoro algebra [14], without using the vertex-face correspondence.

After Smirnov's pioneering works on the sine-Gordon model [3, 15, 16], an axiomatic approach was applied to the XXZ form factors [17, 18, 19], and the  $SU(2)$ -invariant Thirring model [20, 21]. In this paper we wish to address the problem using the same approach.

In Refs. [22, 23], Jackson-type integral formulae were constructed. The relations among those approaches (the vertex operator formalism, the axiomatic approach, and the Jackson integral representation approach) were discussed in Ref. [24]. A rigorous method to obtain correlation functions and form factors was developed in Refs. [25, 26] on the basis of the algebraic Bethe ansatz method. Unfortunately, for technical reasons, this rigorous method has not yet been applied to elliptic integrable models, such as the SOS and XYZ models.

Integral formulae of the XYZ model form factors can be obtained from those of the SOS model form factors by using the vertex-face correspondence. As a preliminary task to be performed for this purpose, we present integral solutions to the quantum Knizhnik-Zamolodchikov equation of level 0 associated with the cyclic SOS model.

The rest of the present paper is organized as follows. In §2 we derive quantum Knizhnik-Zamolodchikov equations of level 0 that form factors in the cyclic SOS model should satisfy. In §3 we construct integral formulae for SOS form factors as solutions to level 0 quantum Knizhnik-Zamolodchikov equations. In §4 we give some concluding remarks. In Appendix A we give a proof of Proposition 1.

## 2 Smirnov's axioms in the SOS model

### 2.1 Boltzmann weights of the cyclic SOS model

The eight-vertex SOS model is a face model [9] which is defined on a square lattice with a site variable  $k_j \in \mathbb{Z}$  attached to each site  $j$ . We call  $k_j$  the local state or height and impose the condition that heights of adjoining sites differ by 1. Local Boltzmann weights are assumed to be functions of the spectral parameter  $\zeta = x^{-u}$  and are denoted by

$$W \left[ \begin{array}{cc|c} c & d & \zeta \\ b & a & \end{array} \right] = W \left[ \begin{array}{cc|c} c & d & u \\ b & a & \end{array} \right],$$

which is defined for a state configuration  $(a, b, c, d)$  ordered clockwise from the SE corner around the face. For fixed  $x$  and  $r$  satisfying  $0 < x < 1$  and  $r > 1$ , the nonzero Boltzmann weights are given as

follows:

$$\begin{aligned} W \left[ \begin{array}{cc|c} k \pm 2 & k \pm 1 & u \\ k \pm 1 & k & \end{array} \right] &= \frac{1}{\kappa(u)}, \\ W \left[ \begin{array}{cc|c} k & k \pm 1 & u \\ k \pm 1 & k & \end{array} \right] &= \frac{1}{\kappa(u)} \frac{[1]\{k \pm u\}}{[1-u]\{k\}}, \\ W \left[ \begin{array}{cc|c} k & k \mp 1 & u \\ k \pm 1 & k & \end{array} \right] &= \frac{1}{\kappa(u)} \frac{[u]\{k \pm 1\}}{[1-u]\{k\}}. \end{aligned} \quad (2.1)$$

Here,  $0 < u < 1$  (Regime III), and we use the following symbols:

$$[u] = x^{\frac{u^2}{r}-u} \Theta_{x^{2r}}(x^{2u}), \quad \{u\} = x^{\frac{u^2}{r}-u} \Theta_{x^{2r}}(-x^{2u}),$$

where

$$\Theta_p(z) := (z; p)_\infty (pz^{-1}; p)_\infty (p, p)_\infty, \quad (a; p_1, \dots, p_n)_\infty = \prod_{k_i \geq 0} (1 - ap_1^{k_1} \dots p_n^{k_n}).$$

The Jacobi theta functions are defined by

$$\theta_1(u; \tau) = \sqrt{-1} q^{\frac{1}{4}} e^{-\sqrt{-1}\pi u} \Theta_{q^2}(e^{2\sqrt{-1}\pi u}), \quad \theta_4(u; \tau) = \Theta_{q^2}(q e^{2\sqrt{-1}\pi u}),$$

where  $q = \exp(\sqrt{-1}\pi\tau)$ . Note that

$$\theta_1\left(\frac{u}{r}; \frac{\pi\sqrt{-1}}{\epsilon r}\right) = \sqrt{\frac{\epsilon r}{\pi}} \exp\left(-\frac{\epsilon r}{4}\right) [u], \quad \theta_4\left(\frac{u}{r}; \frac{\pi\sqrt{-1}}{\epsilon r}\right) = \sqrt{\frac{\epsilon r}{\pi}} \exp\left(-\frac{\epsilon r}{4}\right) \{u\}.$$

The normalization factor

$$\kappa(\zeta) = \zeta^{\frac{r-1}{r}} \frac{(x^4 z; x^4, x^{2r})_\infty (x^2 z^{-1}; x^4, x^{2r})_\infty (x^{2r} z; x^4, x^{2r})_\infty (x^{2r+2} z^{-1}; x^4, x^{2r})_\infty}{(x^4 z^{-1}; x^4, x^{2r})_\infty (x^2 z; x^4, x^{2r})_\infty (x^{2r} z^{-1}; x^4, x^{2r})_\infty (x^{2r+2} z; x^4, x^{2r})_\infty},$$

where  $z = \zeta^2$ , is chosen such that the partition function per site is unity.

The Boltzmann weights (2.1) satisfy the face-type Yang-Baxter equation:

$$\begin{aligned} & \sum_g W \left[ \begin{array}{cc|c} e & f & u_1 - u_2 \\ g & a & \end{array} \right] W \left[ \begin{array}{cc|c} d & e & u_1 \\ c & g & \end{array} \right] W \left[ \begin{array}{cc|c} c & g & u_2 \\ b & a & \end{array} \right] \\ &= \sum_g W \left[ \begin{array}{cc|c} d & e & u_2 \\ g & f & \end{array} \right] W \left[ \begin{array}{cc|c} g & f & u_1 \\ b & a & \end{array} \right] W \left[ \begin{array}{cc|c} d & g & u_1 - u_2 \\ c & b & \end{array} \right]. \end{aligned} \quad (2.2)$$

We can construct  $R(u, \lambda)$  from  $W$  such that  $R(u, \lambda)$  satisfies the dynamical Yang-Baxter equation [27].

## 2.2 Vertex operators and Smirnov's axioms

Let  $p = (k_1, k_2, k_3, \dots)$ , with  $|k_{j+1} - k_j| = 1$  ( $j = 1, 2, 3, \dots$ ), be an admissible path, and let  $\mathcal{H}_{l,k}^{(i)}$  ( $i = 0, 1$ ) be the space of admissible paths satisfying the initial condition  $k_1 = k$  and the following boundary conditions:

$$k_j = \begin{cases} l & \text{if } j \equiv 1 - i \pmod{2}, \\ l + 1 & \text{if } j \equiv i \pmod{2}. \end{cases} \quad (j \gg 1)$$

The type II vertex operator in the SOS model,  $\Psi^{*(1-i,i)}(\zeta)_l^{l'}$  ( $l' = l \pm 1$ ), acts as [10, 11]

$$\Psi^{*(1-i,i)}(\zeta)_l^{l'} : \mathcal{H}_{l,k}^{(i)} \rightarrow \mathcal{H}_{l',k}^{(1-i)} \quad (2.3)$$

and satisfies the commutation relation

$$\Psi^*(\zeta_1)_b^c \Psi^*(\zeta_2)_a^b = \sum_d \Psi^*(\zeta_2)_d^c \Psi^*(\zeta_1)_a^d W' \left[ \begin{array}{cc|c} c & d & \zeta_1/\zeta_2 \\ b & a & \end{array} \right], \quad (2.4)$$

where

$$W' \left[ \begin{array}{cc|c} c & d & \zeta \\ b & a & \end{array} \right] = -W \left[ \begin{array}{cc|c} c & d & \zeta \\ b & a & \end{array} \right] \Big|_{r \mapsto r-1}.$$

In this subsection, we consider form factors in the cyclic SOS model of the following form:

$$F_n^{(i)}(\mathcal{O}; \zeta_1, \dots, \zeta_{2n})_{ll_1 \dots l_{2n-1}l} := \langle \hat{\mathcal{O}} \Psi^{*(i,1-i)}(\zeta_{2n})_{l_{2n-1}}^l \dots \Psi^{*(1-i,i)}(\zeta_1)_l^{l_1} \rangle_i, \quad (2.5)$$

where  $|l_j - l_{j-1}| = 1$  ( $1 \leq j \leq 2n$ ), with  $l_0 = l_{2n} = l$ . Let

$$F_n^{(\sigma)}(\mathcal{O}; \zeta)_{ll_1 \dots l_{2n-1}l} = F_n^{(0)}(\mathcal{O}; \zeta)_{ll_1 \dots l_{2n-1}l} + \sigma F_n^{(1)}(\mathcal{O}; \zeta)_{ll_1 \dots l_{2n-1}l}. \quad (\sigma = \pm 1)$$

Then the following three axioms for the cyclic SOS model hold.

### 1. $W'$ -symmetry

$$\begin{aligned} & F_n^{(\sigma)}(\mathcal{O}; \dots, \zeta_{j+1}, \zeta_j, \dots)_{\dots l_{j-1} l_j l_{j+1} \dots} \\ &= \sum_{l'_j} W' \left[ \begin{array}{cc|c} l_{j+1} & l'_j & \zeta_j/\zeta_{j+1} \\ l_j & l_{j-1} & \end{array} \right] F_n^{(\sigma)}(\mathcal{O}; \dots, \zeta_j, \zeta_{j+1}, \dots)_{\dots l_{j-1} l'_j l_{j+1} \dots} \end{aligned} \quad (2.6)$$

### 2. Cyclicity

$$F_n^{(\sigma)}(\mathcal{O}; \zeta', x^{-2} \zeta_{2n})_{ll_1 \dots l_{2n-1}l} = \sigma F_n^{(\sigma)}(\mathcal{O}; \zeta_{2n}, \zeta')_{l_{2n-1} l l_1 \dots l_{2n-1}}, \quad (2.7)$$

where  $\zeta' = (\zeta_1, \dots, \zeta_{2n-1})$ .

### 3. Annihilation pole condition

$$\begin{aligned} & \text{Res}_{\zeta_{2n} = \varepsilon x^{-1} \zeta_{2n-1}} F_n^{(\sigma)}(\mathcal{O}; \zeta)_{ll_1 \dots l' l} \frac{d\zeta_{2n}}{\zeta_{2n}} = \{l'\}' \left( \delta_{l_{2n-2}l} F_{n-1}^{(\varepsilon\sigma)}(\mathcal{O}; \zeta'')_{ll_1 \dots l_{2n-3}l} \right. \\ & \quad \left. - \sigma \varepsilon \sum_{l'_2 \dots l'_{2n-2}} F_{n-1}^{(\varepsilon\sigma)}(\zeta')_{l' l'_2 \dots l'_{2n-2} l'} \prod_{j=1}^{2n-2} W' \left[ \begin{array}{cc|c} l'_{j+1} & l'_j & \zeta_{2n-1}/\zeta_j \\ l_j & l_{j-1} & \end{array} \right] \right), \end{aligned} \quad (2.8)$$

where  $l_0 = l$ ,  $l'_{2n-1} = l'$  on the RHS of (2.8).

The first two axioms, (2.6) and (2.7), imply the quantum Knizhnik-Zamolodchikov equations of level

0:

$$\begin{aligned}
& F_n^{(\sigma)}(\mathcal{O}; \zeta_1, \dots, x^2 \zeta_j, \dots, \zeta_{2n})_{l_0 l_1 \dots l_{2n-1} l_0} \\
&= \sigma \sum_{l'_1 \dots l'_{j-1} l'_{j+1} \dots l'_{2n}} W' \left[ \begin{array}{cc} l_j & l'_{j-1} \\ l_{j-1} & l_{j-2} \end{array} \middle| \frac{x^2 \zeta_j}{\zeta_{j-1}} \right] \prod_{k=1}^{j-2} W' \left[ \begin{array}{cc} l'_{k+1} & l'_k \\ l_k & l_{k-1} \end{array} \middle| \frac{x^2 \zeta_j}{\zeta_k} \right] \\
&\times W' \left[ \begin{array}{cc} l'_1 & l'_{2n} \\ l_0 & l_{2n-1} \end{array} \middle| \frac{\zeta_j}{\zeta_{2n}} \right] \prod_{k=j+1}^{2n} W' \left[ \begin{array}{cc} l'_{k+1} & l'_k \\ l_k & l_{k-1} \end{array} \middle| \frac{\zeta_j}{\zeta_k} \right] \\
&\times F_n^{(\sigma)}(\mathcal{O}; \zeta_1, \dots, \zeta_j, \dots, \zeta_{2n})_{l'_1 \dots l'_{j-1} l_j l'_{j+1} \dots l'_{2n} l'_1}.
\end{aligned} \tag{2.9}$$

In what follows, we consider only the first two axioms.

### 3 Integral formulae for the cyclic SOS form factors

Let us recall  $z_j = \zeta_j^2 = x^{-2u_j}$ . Set

$$F_n^{(\sigma)}(\mathcal{O}; \zeta)_{l_1 \dots l_{2n-1} l} = c_n \prod_{1 \leq j < k \leq 2n} \zeta_j^{-\frac{r}{r-1}} g^*(z_j/z_k) \times \overline{F}_n^{(\sigma)}(\mathcal{O}; \zeta)_{l_1 \dots l_{2n-1} l}. \tag{3.1}$$

Here,  $c_n$  is a constant, and the function  $g^*(z)$  satisfies

$$\kappa^*(\zeta) = -\kappa(\zeta; x, x^{2(r-1)}) = \zeta^{-\frac{r}{r-1}} \frac{g^*(z)}{g^*(z^{-1})}, \quad g^*(z) = g^*(x^4 z^{-1}). \tag{3.2}$$

The explicit form of  $g^*(z)$  is as follows:

$$g^*(z) = \frac{\{z\}'_\infty \{x^4 z^{-1}\}'_\infty \{x^{2r+2} z\}'_\infty \{x^{2r+6} z^{-1}\}'_\infty}{\{x^2 z\}'_\infty \{x^6 z^{-1}\}'_\infty \{x^{2r} z\}'_\infty \{x^{2r+4} z^{-1}\}'_\infty}, \quad \{z\}'_\infty = (z; x^4, x^4, x^{2(r-1)}). \tag{3.3}$$

In order to present our integral formulae for  $\overline{F}_n^{(\sigma)}(\mathcal{O}; \zeta)$ , let us prepare some notation. Let

$$A_\pm := \{a | l_a = l_{a-1} \pm 1, 1 \leq a \leq 2n\}. \tag{3.4}$$

Then, the number of elements of  $A_\pm$  is equal to  $n$ , because we now set  $l_{2n} = l_0 = l$ . We often use the abbreviated notation  $(w) = (w_{a_1}, \dots, w_{a_n})$  and  $(w') = (w_{a_1}, \dots, w_{a_{n-1}})$  for  $a_j \in A_\pm$  such that  $a_1 < \dots < a_n$ . Let us define the meromorphic function

$$Q_n(w|\zeta)_{l_1 \dots l_{2n-1} l} = \prod_{a \in A_-} \frac{\{v_a - u_a - \frac{3}{2} - l_a\}'}{[u_a - v_a + \frac{3}{2}]'} \left( \prod_{j=1}^{a-1} \frac{[v_a - u_j - \frac{1}{2}]'}{[u_j - v_a + \frac{3}{2}]'} \right) \prod_{\substack{a, b \in A_- \\ a < b}} [v_a - v_b + 1]', \tag{3.5}$$

where  $w_a = x^{-2v_a}$ , and

$$[u]' = x^{\frac{u^2}{r-1}-u} \Theta_{x^{2(r-1)}}(x^{2u}), \quad \{u\}' = x^{\frac{u^2}{r-1}-u} \Theta_{x^{2(r-1)}}(-x^{2u}).$$

Here, we should note that the structure of the expression (3.5) is quite similar to that given in Refs. [20, 24].

We wish to find integral formulae of the form

$$\overline{F}_n^{(\sigma)}(\mathcal{O}; \zeta)_{l_1 \dots l_{2n-1} l} = \prod_{a \in A_-} \oint_{C_a} \frac{dw_a}{2\pi\sqrt{-1}w_a} \Psi_n^{(\sigma)}(w|\zeta) Q_n(w|\zeta)_{l_1 \dots l_{2n-1} l} P_{\mathcal{O}}(w). \quad (3.6)$$

Here,  $P_{\mathcal{O}}(w)$  is an antisymmetric holomorphic function of  $w_{a_1}, \dots, w_{a_n}$ . The kernel has the form

$$\Psi_n^{(\sigma)}(w|\zeta) = \vartheta_n^{(\sigma)}(w|\zeta) \prod_{a \in A_-} \prod_{j=1}^{2n} x^{-\frac{(v_a - u_j)^2}{2(r-1)}} \psi\left(\frac{w_a}{z_j}\right) \prod_{1 \leq j < k \leq 2n} x^{-\frac{(u_j - u_k)^2}{4(r-1)}}, \quad (3.7)$$

where

$$\psi(z) = \frac{(x^{2r+1}z; x^4, x^{2(r-1)})_{\infty} (x^{2r+1}z^{-1}; x^4, x^{2(r-1)})_{\infty}}{(xz; x^4, x^{2(r-1)})_{\infty} (xz^{-1}; x^4, x^{2(r-1)})_{\infty}}. \quad (3.8)$$

For the function  $\vartheta_n^{(\sigma)}(w|\zeta)$ , we assume the following:

- it is anti-symmetric and holomorphic in the variables  $w_a \in \mathbb{C} \setminus \{0\}$ ,
- it is symmetric and meromorphic in the variables  $\zeta_j \in \mathbb{C} \setminus \{0\}$ ,
- it has the two transformation properties

$$\frac{\vartheta_{\sigma}^{(n)}(w|\zeta', x^2\zeta_{2n})}{\vartheta_{\sigma}^{(n)}(w|\zeta)} = \sigma x^{-2n + \frac{2n-1}{r-1}} \prod_{a \in A} w_a^{-1} \prod_{j=1}^{2n} \zeta_j, \quad (3.9)$$

$$\frac{\vartheta_{\sigma}^{(n)}(w', x^4 w_{a_n}|\zeta)}{\vartheta_{\sigma}^{(n)}(w|\zeta)} = x^{-4n} \prod_{j=1}^{2n} \frac{w_{a_n}}{z_j}. \quad (3.10)$$

The function  $\vartheta_n^{(\sigma)}(w|\zeta)$  is otherwise arbitrary, and the choice of  $\vartheta_n^{(\sigma)}(w|\zeta)$  corresponds to that of solutions to (2.6)–(2.8). The transformation properties of  $\vartheta_n^{(\sigma)}(w|\zeta)$  imply

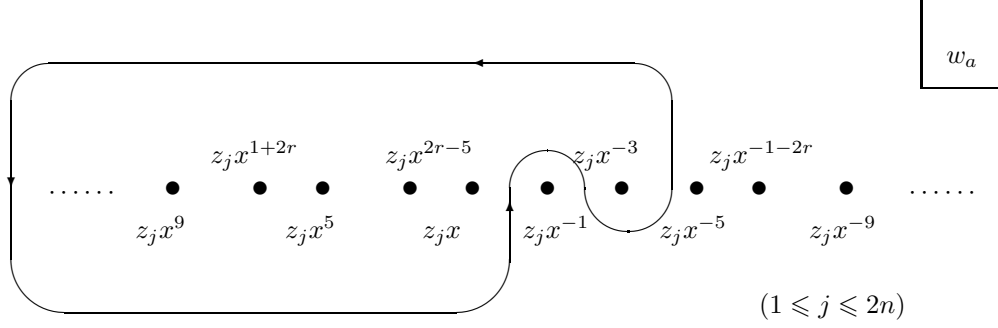
$$\frac{\Psi_n^{(\sigma)}(w|\zeta', x^{-2}\zeta_{2n})}{\Psi_n^{(\sigma)}(w|\zeta)} = \sigma \prod_{j=1}^{2n} \left(\frac{\zeta_j}{\zeta_{2n}}\right)^{\frac{r}{r-1}} \prod_{a \in A_-} \frac{[v_a - u_{2n} - \frac{1}{2}]'}{[u_{2n} - v_a + \frac{3}{2}]'}, \quad (3.11)$$

$$\frac{\Psi_n^{(\sigma)}(w', x^{-4}w_{a_n}|\zeta)}{\Psi_n^{(\sigma)}(w|\zeta)} = \prod_{j=1}^{2n} \frac{[u_j - v_{a_n} - \frac{1}{2}]'}{[v_{a_n} - u_j + \frac{3}{2}]'}. \quad (3.12)$$

The integrand may have poles at the values

$$w_a = \begin{cases} x^{\pm(1+4k_1+2(r-1)k_2)} z_j, & (1 \leq j \leq 2n, k_1, k_2 \in \mathbb{Z}_{\geq 0}) \\ x^{-3+2(r-1)k} z_j. & (1 \leq j \leq 2n, k \in \mathbb{Z}) \end{cases} \quad (3.13)$$

We choose the integration contour  $C_a$  with respect to  $w_a$  ( $a \in A_-$ ) to be along a simple closed curve oriented counter-clockwise that encircles the points  $x^{1+4n_1+2(r-1)n_2} z_j$  ( $1 \leq j \leq 2n, n_1, n_2 \in \mathbb{Z}_{\geq 0}$ ) and  $x^{-3+2(r-1)n_3} z_j$  ( $1 \leq j \leq 2n, n_3 \in \mathbb{Z}_{\geq 0}$ ), but not  $x^{-1-4n_1-2(r-1)n_2} z_j$  ( $1 \leq j \leq 2n, n_1, n_2 \in \mathbb{Z}_{\geq 0}$ ) nor  $x^{-3-2(r-1)n_3} z_j$  ( $1 \leq j \leq 2n, n_3 \in \mathbb{Z}_{>0}$ ). Thus, the contour  $C_a$  actually depends on the variables  $z_j$  in addition to  $a$ , and therefore strictly, it should be written  $C_a(z) = C_a(z_1, \dots, z_{2n})$ . The LHS of (2.7) represents the analytic continuation with respect to  $\zeta_{2n}$ .



We are now in a position to state the following main proposition of the present paper:

**Proposition 1** *Assume the properties of the function  $\vartheta_\sigma^{(n)}$  given below (3.8) and the integration contour  $C_a$  given below (3.13). Then the integral formulae (3.1) and (3.6) with (3.3), (3.5), (3.7) and (3.8) solve the quantum Knizhnik-Zamolodchikov equation (2.9).*

We give a proof of Proposition 1 in Appendix A.

## 4 Concluding remarks

In this paper we have constructed integral formulae for form factors of the cyclic SOS model as solutions to the quantum Knizhnik-Zamolodchikov equation of level 0. It is not clear at this point what kind of local operator corresponds to our solution. This is a general disadvantage of the axiomatic approach.

In our integral formulae, the freedom in the solutions to the quantum Knizhnik-Zamolodchikov equation of level 0 corresponds to the choice of the integral kernel  $\Psi_n^{(\sigma)}(w|\zeta)$ . In order to determine the structure of the space of form factors, we should study the annihilation pole condition for our integral formulae.

It is very important to construct integral formulae for form factors of the XYZ antiferromagnet. If we choose the local operator  $\mathcal{O}$  as the identity operator, this can be done in a manner similar to that in Ref. [1]. In other nontrivial cases this problem is not so easy. In connection with this problem, Lashkevich [8] found a prescription to obtain these integral formulae in the vertex operator formalism. In principle, all form factors corresponding to all local fields can be constructed, but they take very complicated forms. We hope to express XYZ form factors in simpler forms on the basis of the axiomatic approach in a separate paper.

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## A Proof of Proposition 1

In this appendix, we prove (2.6) and (2.7) in order to prove Proposition 1.

### A.1 Proof of $W'$ -symmetry

Let us first prove (2.6). For this purpose, we have to consider four cases, corresponding to  $l_j - l_{j-1} = \pm 1$  and  $l_{j+1} - l_j = \pm 1$ .

Suppose that  $l_j - l_{j-1} = l_{j+1} - l_j = 1$ . Then the relation (2.6) follows from (3.2), because  $j, j+1 \notin A_-$ , and consequently the integrand  $\overline{F}_n^{(\sigma)}(\mathcal{O}; \zeta)$  is symmetric with respect to  $\zeta_j$  and  $\zeta_{j+1}$ .

When  $(l_{j-1}, l_j, l_{j+1}) = (m, m-1, m)$  for some  $m$ , the relation (2.6) reduces to

$$\begin{aligned} & \overline{F}_n^{(\sigma)}(\mathcal{O}; \dots, \zeta_{j+1}, \zeta_j, \dots) \dots mm-1m \dots \\ &= \frac{[1]'\{m - u_j + u_{j+1}\}'}{[1 - u_j + u_{j+1}]'\{m\}'} \overline{F}_n^{(\sigma)}(\mathcal{O}; \dots, \zeta_j, \zeta_{j+1}, \dots) \dots mm-1m \dots \\ &+ \frac{[u_j - u_{j+1}]'\{m-1\}'}{[1 - u_j - u_{j+1}]'\{m\}'} \overline{F}_n^{(\sigma)}(\mathcal{O}; \dots, \zeta_j, \zeta_{j+1}, \dots) \dots mm+1m \dots \end{aligned} \quad (\text{A.1})$$

Note that the set of integration variables in the second term on the RHS is different from that of the other terms. Because the variables  $w_a = x^{-2v_a}$  are the integration variables, we can replace both  $v_j$  and  $v_{j+1}$  in the integrand by  $v$ . Then, the relation (A.1) is obtained by equating the integrands. In this step, we use

$$\begin{aligned} \frac{\{v - u_{j+1} - \frac{1}{2} - m\}'}{[u_{j+1} - v + \frac{3}{2}]'} &= \frac{[1]'\{m - u_j + u_{j+1}\}'}{[1 - u_j + u_{j+1}]'\{m\}'} \frac{\{v - u_j - \frac{1}{2} - m\}'}{[u_j - v + \frac{3}{2}]'} \\ &+ \frac{[u_j - u_{j+1}]'\{m-1\}'}{[1 - u_j - u_{j+1}]'\{m\}'} \frac{\{v - u_{j+1} - \frac{3}{2} - m\}'}{[u_{j+1} - v + \frac{3}{2}]'} \frac{[v - u_j - \frac{1}{2}]'}{[u_j - v + \frac{3}{2}]'}. \end{aligned}$$

Suppose that  $(l_{j-1}, l_j, l_{j+1}) = (m, m+1, m)$  for some  $m$ . The proof in this case can be done in a way similar to that in the previous case. Here we use

$$\begin{aligned} \frac{\{v - u_j - \frac{3}{2} - m\}'}{[u_j - v + \frac{3}{2}]'} \frac{[v - u_{j+1} - \frac{1}{2}]'}{[u_{j+1} - v + \frac{3}{2}]'} &= \frac{[u_j - u_{j+1}]'\{m+1\}'}{[1 - u_j - u_{j+1}]'\{m\}'} \frac{\{v - u_j - \frac{1}{2} - m\}'}{[u_j - v + \frac{3}{2}]'} \\ &+ \frac{[1]'\{m + u_j - u_{j+1}\}'}{[1 - u_j + u_{j+1}]'\{m\}'} \frac{\{v - u_{j+1} - \frac{3}{2} - m\}'}{[u_{j+1} - v + \frac{3}{2}]'} \frac{[v - u_j - \frac{1}{2}]'}{[u_j - v + \frac{3}{2}]'}. \end{aligned}$$

Finally, let  $(l_{j-1}, l_j, l_{j+1}) = (m, m-1, m-2)$  for some  $m$ . Then the integrand on the RHS of (2.6) contains the factor

$$\begin{aligned} & I(u_j, u_{j+1}; v_j, v_{j+1}) \\ &= \frac{\{v_j - u_j - \frac{1}{2} - m\}'}{[u_j - v_j + \frac{3}{2}]'} \frac{\{v_{j+1} - u_{j+1} + \frac{1}{2} - m\}'}{[v_{j+1} - u_{j+1} + \frac{3}{2}]'} \frac{[v_{j+1} - u_j - \frac{1}{2}]'}{[u_j - v_{j+1} + \frac{3}{2}]'} [v_j - v_{j+1} + 1]'. \end{aligned}$$

The corresponding factor on the LHS should be equal to  $I(u_{j+1}, u_j; v_j, v_{j+1})$ . Thus, the difference between the two sides contains the factor

$$\begin{aligned} & I(u_j, u_{j+1}; v_j, v_{j+1}) - I(u_{j+1}, u_j; v_j, v_{j+1}) = \{m-1\}' \times \\ & \times [v_j - v_{j+1} - 1]' [v_j - v_{j+1} + 1]' [u_j - u_{j+1}]' \{v_j + v_{j+1} - u_j - u_{j+1} - m - 1\}', \end{aligned}$$

which is symmetric with respect to  $w_j = x^{-2v_j}$  and  $w_{j+1} = x^{-2v_{j+1}}$ . Because  $\Psi_n^{(\sigma)}(w|\zeta)$  is antisymmetric with respect to the variables  $w_a$ , the relation (2.6) in this case does hold.

## A.2 Proof of the cyclicity

In the proof of the cyclicity (2.7) we have to consider the two cases  $l_{2n-1} = l \pm 1$ . First, let  $l_{2n-1} = l - 1$ . Note that  $F_n^{(\sigma)}(\mathcal{O}; \zeta)_{l \dots l-1l}$  actually has no pole at the point  $w_a = x^{-3}z_{2n}$ , because  $2n \notin A_-$ . When the integral (3.6) is analytically continued from  $\zeta_{2n} = x^{-u_{2n}}$  to  $x^{-2}\zeta_{2n} = x^{-(u_{2n}+2)}$ , the points  $x^{1+4n_1+2(r-1)n_2}z_{2n}$  and  $x^{-1-4n_1-2(r-1)n_2}z_{2n}$  ( $n_1, n_2 \in \mathbb{Z}_{\geq 0}$ ) move to the points  $x^{-3+4n_1+2(r-1)n_2}z_{2n}$  and  $x^{-5-4n_1-2(r-1)n_2}z_{2n}$ , respectively. Thus, the integral contour  $C'_a = C_a(z', x^{-4}z_{2n})$  coincides with the original one:  $C_a = C_a(z)$ .

Furthermore, using (3.11) we obtain

$$\Psi_n^{(\sigma)}(w|\zeta', x^{-2}\zeta_{2n}) = \sigma \Psi_n^{(\sigma)}(w|\zeta) \prod_{j=1}^{2n-1} \left( \frac{\zeta_j}{\zeta_{2n}} \right)^{\frac{r}{r-1}} \prod_{a \in A_-} \frac{[v_a - u_{2n} - \frac{1}{2}]'}{[u_{2n} - v_a + \frac{3}{2}]'},$$

which implies that the integrands of the two sides of (2.7) are identified, and therefore the relation (2.7) holds when  $l_{2n-1} = l - 1$ .

Next, let  $l_{2n-1} = l + 1$ . In this case we rescale the variable  $w_{2n}$  as  $w_{2n} \mapsto x^{-4}w_{2n}$  ( $v_{2n} \mapsto v_{2n} + 2$ ) on the LHS of (2.7). Then, the integral contour with respect to  $w_a$  ( $a \in A_- \setminus \{2n\}$ ) will be  $C'_a = C_a(z', x^{-4}z_{2n}) = C_a(z)$ , for the same reason as in the previous case. The other contour,  $\tilde{C} = C_{2n}(x^4z', z_{2n})$ , encircles  $x^{5+4n_1+2(r-1)n_2}z_j$ ,  $x^{1+2(r-1)n_3}z_j$  and  $x^{1+4n_1+2(r-1)n_2}z_{2n}$ ,  $x^{-3+2(r-1)n_3}z_{2n}$ , but not  $x^{3-4n_1-2(r-1)n_2}z_j$ ,  $x^{1-2(r-1)(n_3+1)}z_j$  nor  $x^{-1-4n_1-2(r-1)n_2}z_{2n}$ ,  $x^{-3-2(r-1)(n_3+1)}z_{2n}$ , where  $1 \leq j \leq 2n-1$  and  $n_1, n_2, n_3 \in \mathbb{Z}_{\geq 0}$ .

Since  $Q^{(n)}(w', x^{-4}w_{2n}|\zeta', x^{-2}\zeta_{2n})_{l \dots l+1l}$  contains the factor

$$\begin{aligned} & J(w', x^4w_{2n}|\zeta', x^2\zeta_{2n}) \\ &= \frac{\{v_{2n} - u_{2n} - \frac{3}{2} - l\}'}{[u_{2n} - v_{2n} + \frac{3}{2}]'} \prod_{j=1}^{2n-1} \frac{[v_{2n} - u_j + \frac{3}{2}]'}{[u_j - v_{2n} - \frac{1}{2}]'} \prod_{\substack{a \in A_- \\ a \neq 2n}} [v_a - v_{2n} - 1]', \end{aligned} \quad (\text{A.2})$$

the pole at  $w_{2n} = x^3z_j$  ( $1 \leq j \leq 2n-1$ ) disappears. Thus, we can deform the contour  $\tilde{C}$  so as to coincide with the original one,  $C_{2n} = C_{2n}(z)$ , without crossing any poles. Thus, the integral contours on the two sides of (2.7) coincide.

Next, replace the integral variables as  $(w', w_{2n}) \mapsto (w_{2n}, w')$  on the RHS of (2.7) and compare the integrands of the two sides. Then, the corresponding factor on the RHS of (A.2) is equal to

$$J(w_{2n}, w'|\zeta) = \frac{\{v_{2n} - u_{2n} - \frac{3}{2} - l\}'}{[u_{2n} - v_{2n} + \frac{3}{2}]'} \prod_{\substack{a \in A_- \\ a \neq 2n}} \frac{[v_a - u_{2n} - \frac{1}{2}]'}{[u_{2n} - v_a + \frac{3}{2}]'} [v_{2n} - v_a + 1]'. \quad (\text{A.3})$$

From (A.2), (A.3), (3.11), (3.12) and the antisymmetric property of  $\Psi_n^{(\sigma)}$  with respect to the variables  $w_a$ , we have

$$\begin{aligned} & \Psi_n^{(n)}(w', x^{-4}w_{2n}|\zeta', x^{-2}\zeta_{2n}) J(w', x^{-4}w_{2n}|\zeta', x^{-2}\zeta_{2n}) \\ &= \sigma \Psi_n^{(\sigma)}(w_{2n}, w'|\zeta) J(w_{2n}, w'|\zeta) \prod_{j=1}^{2n-1} \left( \frac{\zeta_j}{\zeta_{2n}} \right)^{\frac{r}{r-1}}, \end{aligned} \quad (\text{A.4})$$

which implies that the integrands of the two sides of (2.7) are identical, and therefore the relation (2.7) holds when  $l_{2n-1} = l + 1$ .

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